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Sec — C

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~~for~~, Taylor's Series Method

Consider the Ist order diff. eq^n

$$y' = f(x, y) \quad \text{--- (1)}$$

Here $y' = \frac{dy}{dx}$.

Let the initial conditions be given as

$y = y_0$ at $x = x_0$.

Let $y = f(x)$ be the sol^n of the eq^n (1)
Let's expand y i.e. $y = f(x)$ as in Taylor's series around $x = x_0$, then we get

$$y = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{\lfloor 2} f''(x_0)$$

$$+ \frac{(x-x_0)^3}{\lfloor 3} f'''(x_0) + \text{———}$$

$$\frac{(x-x_0)^n}{\lfloor n} f^n(x_0)$$

or

In this eq^n

$$y_0 = f(x_0), \quad y_0' = f'(x_0)$$

$$y_0'' = f''(x_0) \text{ and so on}$$

$$\begin{cases} y(x_0) = y_0 & \text{i.e. } y(0)=1 \\ y_0 = 1, \; x_0 = 0 \end{cases}$$

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$$y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{\underline{L2}} y_0'' + \frac{(x-x_0)^3}{\underline{L3}} y_0''' + \text{———}$$

Ex- obtain the solution of $y' = 3x + y^2$ using Taylor's series method $y = 1$ when $x = 0$ [DI means $y(x_0) = 1$]
find the value of y at $x = 0.1$
[~~DI mean~~

Sol^n:-

$$y' = \frac{dy}{dx} = 3x + y^2$$

$$y'' = 3 + 2yy'$$

$$y''' = 0 + 2[y \cdot y'' + y'y'] = 2yy'' + 2y'y'$$

$$y^{(4)} = 2yy''' + 2y'y'' + 2y'y'' + 2y'y'' = 2yy''' + 6y'y''$$

It is given that initially, $x_0 = 0$, $y_0 = 1$
putting these values in above eq^n, we get

$y_0' = y' = 3\times 0 + (1)^2 = 1 \quad \therefore y_0' = 1$

$y_0'' = y'' = 3 + 2\times 1\times 1 = 5 \quad \therefore y_0'' = 5$

$y_0''' = y''' = 2\times 1\times 5 + 2\times 1\times 1 = 12, \quad y_0''' = 12$

$y_0^{(4)} = y^{(4)} = 2\times 1\times 12 + 6\times 1\times 5 = 54, \quad y_0^{(4)} = 54$

By using Taylor Series

$$y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{\lfloor 2} y_0'' + \frac{(x-x_0)^3}{\lfloor 3} y_0''' + \frac{(x-x_0)^4}{\lfloor 4} y_0^{(4)} + \ldots$$

putting the values in the above eqn.

$$y = 1 + (x-0)(1) + \frac{(x-0)^2}{2}(5) + \frac{(x-0)^3}{\cancel{6}}(\cancel{12}^{2}) + \frac{(x-0)^4}{\cancel{24}_{4}}(\cancel{54}^{9})$$

$$\boxed{y = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4} \quad —(1)$$

Find the value of y at $x = 0.1$

$\therefore$ Putting the value of $x = 0.1$ in above eqn.

$$y = 1 + 0.1 + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \frac{9}{4}(0.1)^4$$

$$= \boxed{1.127225} \quad \text{Ans}$$

Note for further calculate we assume initial values $x_0 = 0.1$ and $y_0 = 1.127225$

Euler's method

Euler's method is used to find an approximate solution of ordinary differential equations.

Consider the differential equation

$$\frac{dy}{dx} = f(x, y),$$

when $y = y_0$ at $x = x_0$

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

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Ex- Apply Euler's method to solve $\frac{dy}{dx} = x+y$, $y(0) = 0$, choosing the step length $= 0.2$ find $y(1.4)$.

Sol^n :-

we have $\frac{dy}{dx} = x+y$.

or

$f(x,y) = x+y$ and $h = 0.2$

Consider initial values as $x_0 = 0$, $y_0 = 0$

The approximate value of y at $x_0 = 0$

$y_0 = 0.$

The approximate value of y at $x_1 = 0.2$

$y_1 = y_0 + h f(x_0, y_0)$

$= 0 + 0.2(x_0 + y_0)$

$= 0 + 0.2(0+0) = \underline{0}$

The approximate value of y at $x_2 = 0.4$.

$y_2 = y_1 + h f(x_1, y_1)$

$= 0 + 0.2(x_1 + y_1) = 0 + 0.2[0.2 + 0]$

$\underline{0.04}$

The approximate value of y at $x_3 = 0.6$

$y_3 = y_2 + h f(x_2, y_2)$

$= 0.04 + 0.2 (x_2 + y_2)$

$= 0.04 + 0.2 (0.4 + 0.04)$

$= 0.128.$

The approximate value of y at $x_4 = 0.8.$

$y_4 = y_3 + h f(x_3, y_3)$

$= 0.128 + 0.2 (x_3 + y_3)$

$= 0.128 + 0.2 (0.6 + 0.128)$

$= 0.2736.$

The approximate value of y at $x_5 = 1.0$

$y_5 = y_4 + h f(x_4, y_4)$

$= y_4 + h (x_4 + y_4)$

$= 0.2736 + 0.2 (0.8 + 0.2736)$

$= 0.4883.$

The approximate value of y at $x_6 = 1.2$

$y_6 = y_5 + h f(x_5, y_5)$

$= 0.4883 + 0.2 (x_5 + y_5)$

$= 0.4883 + 0.2 (1.0 + 0.4883)$

$= 0.786$

The approximate value of y at $x_7 = 1.4$

$$y_7 = y_6 + h f(x_6, y_6)$$

$$= 0.786 + 0.2(x_6 + y_6)$$

$$= 0.786 + 0.2(1.2 + 0.786)$$

$$= 1.1882$$

Hence :
$y(1.4) = 1.1832$ approximately

Note -

$$f(x,y) = x+y$$

$$f(x,y) = x^2 - 2y$$

$$f(x_6, y_6) = [x_6^2 - 2y_6]$$

⇒ Euler's modified form :-
If we calculate
$y_1 = y_0 + h f(x_0, y_0) = 0 + 0.2(0+0) = 0$

⇒ By Euler's ~~method~~ modified method, the approx. to y_1 are -

$$y_1^{(1)} = y_0 + \frac{h}{2} f(x_0, y_0) + f(x_1, y_1)$$

$$= y_0 + \frac{h}{2} [-x_0 y_0^2 - x_1 y_1^2]$$

Use it at each step

Runge Kutta Method

Runge kutta method gives greater accuracy as compared to the other methods which are used to solve the differential equations by finding out the values of y at any point x to be considered.

Ist of order — Euler's method.

R.K. method of order. 2

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = \cancel{h f(x_0, y_0)} \quad h f(x_0 + h, y_0 + k_1)$$

3 order :-

$$y_{n+1} = y_n + \frac{1}{6} h [k_1 + 4k_2 + k_3]$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + h/2, y_n + \frac{k_1 \cdot h}{2}\right)$$

$$k_3 = f(x_n + h, y_n - k_1 h + 2k_2 h)$$

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R. K. method of order 4.

$$\boxed{y_{n+1} = y_n + \frac{1}{6} h\left[k_1 + 2k_2 + 2k_3 + k_4\right]}$$

$k_1 = h f(x_0, y_0)$ i.e. $h f(x_n, y_n)$.

(h is not allowed)

$k_2 = h f(x_0 + h/2, y_0 + k_1/2)$.

$k_3 = h f(x_0 + h/2, y_0 + k_2/2)$

$k_4 = h f(x_0 + h, y_0 + k_3]$

OR we can also write.

$K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_1 = y_0 + k$

$$\boxed{y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]}$$

h is allowed while finding the values of k_1, k_2, k_3, and k_4 etc

Ex:- Apply Runge kutta fourth order method to find an approximate value of y when x = 0.2 i.e. at x = 0.2

Given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0, i.e. y(0) = 1

Sol^n :-

The given differential equation is

$$\frac{dy}{dx} = f(x,y) = x+y.$$

The initial conditions are y(0) = 1 i.e

$x_0 = 0,\ y_0 = 1$

Taking h = 0.2, we have,

$$k_1 = h f(x_0, y_0) = h(x_0 + y_0) = 0.2 \times (0+1)$$
$$= 0.2$$
$$= 0.2$$

$$k_2 = h f(x_0 + h/2,\ y_0 + k_1/2) =$$
$$= h\left[(x_0 + h/2) + (y_0 + k_1/2)\right]$$
$$= 0.2\left[(0 + 0.2/2) + \left(1 + \frac{0.2}{2}\right)\right]$$
$$= 0.2\left[0.1 + 1.1\right] = 0.24$$

$$k_2 = 0.24$$

$$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2)$$

$$= k_3 = 0.2\left[(x_0 + h/2) + \left(y_0 + \frac{k_2}{2}\right)\right]$$

$$= 0.2\left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.24}{2}\right)\right]$$

$$= 0.2[(0.1) + 1.12]$$

$$= 0.2 \times 1.22$$

$$= 0.2440$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3)$$

$$= h[(x_0 + h) + (y_0 + k_3)]$$

$$= 0.2[(0 + 0.2) + (1 + 0.2440)]$$

$$= 0.2[0.2 + 1.2440]$$

$$= 0.2[0.2 + 1.244]$$

$$= 0.2888.$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 0.24 \times 2 + 2 \times 0.244 + 0.2888)$$

$$= \frac{1}{6}(0.2 + 0.48 + 0.488 + 0.2888)$$

$$= 0.2428.$$

$$y_1 = y_0 + k = 1 + 0.2428 = \boxed{1.2428}$$

Required approximate value of y at $x = 0.2$ is

$\boxed{y = 1.2428}$ Ans

at $x = 0.1, 0.2, 0.3, 0.4$ $y_2 = x_1 = 0.1$, $h =$ always $-$ 0.1 — $y_1 = x_0 = 0, y_0 = \ldots$

$y_1 =$ obtained value. similarly $y_2 = x_2$, $y_2 = ?$

[Continue find values separately]

Numerical :-

Use the Runge Kutta method to solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, for the interval $0 < x \leq 0.4$, with $h = 0.1$

Sol^n :- Given diff. eq^n —

$$10\frac{dy}{dx} = x^2 + y^2$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{10} = f(x, y)$$

Take $h = 0.1$,

at initial conditions $= x_0 = 0,\ y_0 = 1$

$x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4$

$y_1 =$, we have $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$k_1 = h f(x_0, y_0) = ?$

$k_2 = ?$

$k_3 = ?$

$k_4 = ?$

$\boxed{y_1 = y_0 + k} = \quad y_1 = ?$

To compute y_2 we ~~have~~ use

$x_1 = 0.1$, 0.1, $y_1 = ?$ i.e

obtained from above

and $h = 0.1$

$k_1 = hf(x_1, y_1) = ?$

$k_2 = - - -$

$k_3 - -$

$k_4 - - -$

Similarly To compute y_3,

we use $x_2 = 0.2$, $y_2 = ?$

$h = 0.1$

$k_1 = hf(x_2, y_2)$.

$k_2 = - - -$

$k_3 - - -$

$- -$

Similarly

To comp $y_4 = x_3 = 0.3$, $y_3 = ?$

$h = 0.1$

$k_1 = hf(x_3, y_3)$.

$$\boxed{y_4 = y_3 + K}$$

Hence the largest eigen value is 4.98, approximately 5.

Example 3. *Using power method find the largest eigen value and the corresponding eigen vector of the matrix* $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$

Solution. Let the initial eigen vector be $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then

$$AX_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix}$$

$\therefore$ The first approximation to the eigen value is 6 and the corresponding eigen vector is

$$X^{(1)} = \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix}$$

Hence
$$AX^{(1)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.166 \\ 2.336 \\ 8.003 \end{bmatrix} = 8.003\begin{bmatrix} 0.0208 \\ 0.292 \\ 1 \end{bmatrix}$$

$\therefore$ The second approximation to the eigen value is 8.003 and the corresponding eigen vector is

$$X^{(2)} = \begin{bmatrix} 0.21 \\ 0.292 \\ 1 \end{bmatrix}$$

Repeating the above process, we get

$$AX^{(2)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.21 \\ 0.292 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.145 \\ 0.252 \\ 6.002 \end{bmatrix} = 6.002\begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix}$$

$$AX^{(3)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.065 \\ -0.068 \\ 6.272 \end{bmatrix} = 6.272\begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix}$$

$$AX^{(4)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.362 \\ 0.283 \\ 6.941 \end{bmatrix} = 6.941\begin{bmatrix} 0.340 \\ 0.185 \\ 1 \end{bmatrix}$$

Hence the largest eigen value is 6.941 and the corresponding eigen vector is $\begin{bmatrix} 0.340 \\ 0.185 \\ 1 \end{bmatrix}$.